U S A Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 4 - Year 10 - Academic Year 1998-99

Gene A. Berg, Editor (phone 301-688-3175)

1/4/10. Exhibit a 13-digit integer N that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

Solution 1 by Ann Marie Cody (12/MA): Let N = ABCDEFGHIJKLM, where A, B, ..., M are digits.

N is divisible by 2, so M must be even. Therefore M = 8.

N = ABCDEFGHIJKx100 + LM and is divisible by 2^{13}. N and ABCDEFGHIJKx100 are each divisible by 4, so LM must be divisible by 4. LM = L8, 88 is divisible by 4, and 98 is not, so LM = 88.

N = ABCDEFGHIJx1000 + KLM is divisible by 8 and 1000 is divisible by 8, so KLM must be divisible by 8. 8 does not divide 988, so KLM = 888.

N = ABCDEFGHIx10,000 + JKLM. N and 10,000 are divisible by 2^4, so JKLM is also. JKLM = J888. 8888 is not divisible by 16 but 9888 is, so JKLM = 9888.

N = ABCDEFGHx100,000 + IJKLM. N and 100,000 are divisible by 2^5, so IJKLM is also. IJKLM = I9888. 99888 is not divisible by 32 but 89888 is, so IJKLM = 89888.

N = ABCDEFGx1,000,000 + HIJKLM. N and 1,000,000 are divisible by 2^6, so HIJKLM is also. HIJKLM = H89888. 889888 is not divisible by 64 but 989888 is, so HIJKLM = 989888.

N = ABCDEFx10,000,000 + GHIJKLM. N and 10,000,000 are divisible by 2^7, so GHIJKLM is also. GHIJKLM = G989888. 8989888 is not divisible by 128 but 9989888 is, so GHIJKLM = 9989888.

N = ABCDEx100,000,000 + FGHIJKLM. N and 100,000,000 are divisible by 2^8, so FGHIJKLM is also. FGHIJKLM = F9989888. 99989888 is not divisible by 256 but 89989888 is, so FGHIJKLM = 89989888.

N = ABCDx10^9 + EFGHIJKLM. N and 10^9 are divisible by 2^9, so EFGHIJKLM is also. EFGHIJKLM = E89989888. 889989888 is not divisible by 512 but 989989888 is, so EFGHIJKLM = 989989888.

N = ABCx10^{10} + DEFGHIJKLM. N and 10^{10} are divisible by 2^{10}, so DEFGHIJKLM is also. DEFGHIJKLM = D989989888. 9989989888 is not divisible by 2^{10} but 8989989888 is, so DEFGHIJKLM = 8989989888.

N = ABx10^{11} + CDEFGHIJKLM. N and 10^{11} are divisible by 2^{11}, so CDEFGHIJKLM is also. CDEFGHIJKLM = C8989989888. 88989989888 is not divisible by 2^{11} but 98989989888 is, so CDEFGHIJKLM = 98989989888.

N = Ax10^{12} + BCDEFGHIJKLM. N and 10^{12} are divisible by 2^{12}, so BCDEFGHIJKLM is also. BCDEFGHIJKLM = B98989989888. 998989989888 is not divisible by 2^{12} but 898989989888 is, so BCDEFGHIJKLM = 898989989888.

N and 10^{13} are divisible by 2^{13}. N=ABCDEFGHIJKLM = A98989989888.
9898989989888 is not divisible by 2^{13} but 8898989989888 is, so
N = 8898989989888

Solution 2 by Oaz Nir (10/CA): Answer: 8898989989888

Proof: More generally, we prove that for all positive integers n and non-zero digits a and b, with a even and b odd, there exists an integer x_n with exactly n digits that is an integer multiple of 2^n, whose digits consist of only a's and b's. The proof is by mathematical induction. For the base case we take $x_1 = a$. Clearly this number is divisible by $2^1 = 2$, as required. For the induction step, we assume that an integer x_n exists with the desired properties, and we show how to construct x_{n+1}. We need two cases:

Case 1: x_n is divisible by 2^{n+1}.

Take x_{n+1} equal to x_n with the digit a appended to the left. That is, $x_{n+1} = a \cdot 10^n + x_n$.

It is easy to see that x_{n+1} is divisible by 2^{n+1} (because both terms $a \cdot 10^n$ and x_n are).

Case 2: x_n is not divisible by 2^{n+1}.

Take x_{n+1} equal to x_n with the digit b appended to the left. That is, $x_{n+1} = b \cdot 10^n + x_n$.

It is easy to see that x_{n+1} is divisible by 2^{n+1} (because both terms $b \cdot 10^n$ and x_n are odd multiples of 2^n, so their sum is an even multiple of 2^n, and is therefore a multiple of 2^{n+1} as required).

This completes the proof of the general result. We can use our method of proof to actually calculate the numbers x_n. In particular, when $a = 8$, $b = 9$, and $n = 13$, we proceed as follows: $x_1 = 8$, $x_2 = 88$, $x_3 = 888$, $x_4 = 9888$, $x_5 = 89888$, $x_6 = 989888$, $x_7 = 9989888$, $x_8 = 89989888$, $x_9 = 989989888$, $x_{10} = 8989989888$, $x_{11} = 98989989888$, $x_{12} = 898989989888$, and $x_{13} = 8898989989888$, where the numbers x_2, x_3, x_5, x_8, x_{10}, x_{12}, and x_{13} were formed as in Case 1, and the numbers x_4, x_6, x_7, x_9, and x_{11}, were formed as in Case 2.

Editor's comments: We congratulate Mr. Nir for winning the Brilliancy Award from the Bay Area Mathematical Olympiad for a very elegant solution. The BAMO is a regional mathematical competition in the San Francisco area.

Solution 3 by Zhihao Liu (9/IL): Answer: 8898989989888

N is divisible by 2^{13}. Therefore for every integral value of k from 1 to 13, the last k digits of N are divisible by 2^k. Working from right to left, the unit, tens, and hundreds digits must all be 8, since 888 is divisible by 2^3.

Also, if a k-digit number is divisible by 2^k, then when it is divided by 2^{k+1}, the remainder is either 0 or 2^k. Using this fact, we can deduce the other digits of N. Since
$888 \equiv 8 \equiv 9000 \bmod 16$, thus $9000 + 888 \equiv 8 + 8$, and 9888 is a multiple of 16. Similarly, $9888 \equiv 0 \bmod 32$, hence 32 divides 89888. Performing this algorithm eight more times:
$$89888 \equiv 32 \bmod 64$$

$$\Rightarrow 989888 \equiv 64 \bmod 128$$
$$\Rightarrow 9989888 \equiv 0 \bmod 256$$
$$\Rightarrow 89989888 \equiv 256 \bmod 512$$
$$\Rightarrow 989989888 \equiv 0 \bmod 1024$$
$$\Rightarrow 8989989888 \equiv 1024 \bmod 2048$$
$$\Rightarrow 98989989888 \equiv 0 \bmod 2048$$
$$\Rightarrow 898989989888 \equiv 0 \bmod 4096$$
$$\Rightarrow 8898989989888$$

Indeed, 8898989989888 is divisible by 2^{13} as it can be expressed as 8192x1086302489. Therefore, **N = 8898989989888**.

Editor's comments on modular arithmetic: This may be a good opportunity to briefly discuss modular arithmetic as used in Mr. Liu's solution above. Instead of the standard clock with twelve digits, *1, 2, 3, ..., 12*, consider a clock that has *p* digits, *0, 1, 2, ..., p-1*. As a specific example, consider a clock that has *p* = seven digits, *0, 1, 2, 3, 4, 5, 6*. With this clock we **add** as follows: *4 + 5* corresponds to the time starting at four o'clock and advancing five hours to nine o'clock, which on this clock is *2* o'clock. Thus, on this clock *4 + 5 = 2*. We write this $4 + 5 \equiv 2$, or to make sure everyone knows we are doing clock arithmetic on a clock with *7* digits, we might write $4 + 5 \equiv 2 \bmod 7$. We read this expression as *four plus five is congruent to 2 modulo seven*. Similarly, $6 + 3 \equiv 2$, $4 + 3 \equiv 0$, etc. Notice in each case, we only care what the remainder is after division by seven; in the example *4 + 5*, notice *4 + 5 = 9*, and *9* divided by *7* is *1* with a *remainder* of *2*, so $4 + 5 \equiv 2 \bmod 7$. We don't seem to care what the quotient is, we only care about the remainder after division by seven. We can also **subtract** with clock arithmetic: *4 - 5* corresponds to starting at four o'clock and backing up five hours, which on this clock is *6* o'clock. Thus, $4 - 5 \equiv 6$. Observe that $4 - 4 \equiv 4 + 3 \equiv 0$, so we may think of *3* as *-4*; that is the additive inverse of *4* is *3* [i.e. three is the number we must add to four to get zero]. Knowing how to add allows us to **multiply**: notice that 3×5 corresponds to adding 5 three times, and we already know how to add, so $3 \times 5 = 5 + 5 + 5 \equiv 1$. With this idea we can now multiply in this clock arithmetic system. Other examples of multiplication are $6 \times 3 \equiv 4$, $4 \times 3 \equiv 5$, etc. So now we can add, subtract, and multiply on this clock. **Division** is more difficult. If our clock has a prime number of digits, then we can always divide by nonzero numbers. But if our clock has *p* digits where *p* is not prime, we may not always be able to divide by every nonzero number. In our example with *p = 7* observe that $3 \times 5 = 5 + 5 + 5 \equiv 1$, so five is the multiplicative inverse of three [i.e. five is the number we must multiply by three to get one]. We write $\frac{1}{3} \equiv 5 \bmod 7$. Similarly, $\frac{1}{2} \equiv 4 \bmod 7$, $\frac{1}{4} \equiv 2 \bmod 7$, etc. Just to make sure you understand, before reading ahead, find $\frac{1}{6}$; that is, find the number you must multiply by *6* to get *1* modulo *7*.

When *p* is a prime, Euclid's algorithm helps us find multiplicative inverses. When *p* is *7* it is easy to do this just by guessing and trying. But if *p* is large, say *p = 163* or an even larger prime number, it is more difficult to guess and check. With *p = 163* we use the Euclidean algorithm as in the following example. Suppose we have the number 25 and we wish to find the multiplicative

inverse of *25* modulo *163*. That is, we wish to find an integer k such that $25 \times k \equiv 1 \bmod 163$. The Euclidean algorithm gives us the greatest common divisor of *25* and *163*. Since *163* is prime and 0<25<163, we know this will be one. The process of the Euclidean algorithm gives us more information, which solves our problem. This process is sometimes called the extended Euclidean algorithm:

Divide 25 into 163 to get quotient 6 and remainder 13. $13 = 163 - 6 \times 25$.

Divide 13 into 25 to get quotient 1 and remainder 12. $12 = 25 - 1 \times 13$.

Divide 13 by 12 to get quotient 1 and remainder 1. $1 = 13 - 1 \times 12$.

Now back up through this process.

$$1 = 13 - 1 \times 12$$
$$1 = (1 \times 13) - 1 \times (25 - 1 \times 13) = (2 \times 13) - (1 \times 25)$$
$$1 = 2 \times (163 - 6 \times 25) - (1 \times 25)$$
$$1 = (2 \times 163) - (13 \times 25)$$

Thus, the greatest common divisor of 25 and 163 is one, and can be written as in the last equation above. Since I only care about the remainder after division by 163, and the term 2×163 will contribute zero to this remainder, I can write

$$1 \equiv -13 \times 25 \bmod 163$$

Since $-13 \equiv 150 \bmod 163$, we rewrite this as

$$1 \equiv 150 \times 25 \bmod 163.$$

So the multiplicative inverse of *25* modulo *163* is *150*. With clock arithmetic on a clock with $p = 163$, division by *25* is equivalent to multiplication by *150*.

Again, when p is prime we are guaranteed that each nonzero digit will have a multiplicative inverse, so we can divide by all nonzero digits. When p is not a prime we can still add, subtract, and multiply modulo p, but we cannot always divide by nonzero numbers because on these clocks there are nonzero divisors of zero.

$\dfrac{1}{6} \equiv 6 \bmod 7$ is the answer to the exercise posed above. Demonstrate it using Euclid's Algorithm.

Editor's comments: Thanks go to Professor George Berzsenyi, the creator of the USAMTS and our continuing supporter, for posing this interesting problem. Round 4 completes the tenth year of the USAMTS.

2/4/10. For a nonzero integer i, the exponent of 2 in the prime factorization of i is called $ord_2(i)$. For example, $ord_2(9) = 0$ since 9 is odd, and $ord_2(28) = 2$ since $28 = 2^2$ x 7. The numbers 3^n - *1* for n = *1, 2, 3,...* are all even, so $ord_2(3^n - 1) > 0$ for $n > 0$.

a) For which positive integers n is $ord_2(3^n - 1) = 1$?

b) For which positive integers n is $ord_2(3^n - 1) = 2$?

c) For which positive integers n is $ord_2(3^n - 1) = 3$?

Prove your answers.

Solution 1 by Aaron Marcus (10/TX): (a.) $3^n - 1$ can be factored into
$(3-1)(1+3+3^2+...+3^{n-1})$. Since the second factor consists of only odd elements, the sum will be odd if and only if there are an odd number of elements. Since there are $(n-1)+1=n$ elements (exponents 0 through $n-1$), $ord_2(3^n - 1) = 1$ when n is odd, or when $n = 2k + 1$.

    (b.) Consider when n is even. The terms in the second factor can be paired. Thus,

$$3^n - 1 = (3-1)(1+3+3^2+...+3^{n-1})$$
$$= 2((1+3\cdot1)+(3^2+3\cdot3^2)+...(3^{n-2}+3\cdot3^{n-2}))$$
$$= 2((4\cdot1)+(4\cdot3^2)+...(4\cdot3^{n-2}))$$
$$= 8(1+3^2+...3^{n-2})$$

Since n can only be even or odd, there is no case where $ord_2(3^n - 1) = 2$ (it must be either 1 or greater than or equal to 3).

    (c.) From above, for $ord_2(3^n - 1) = 3$, $(1+3^2+...+3^{n-2})$ must be odd.

$1+3^2+...+3^{n-2} = 1+9+9^2+...+9^{\frac{n}{2}-1}$. Since all terms are odd, this can only be odd if there are an odd number of terms. There are $\frac{n}{2}$ terms in the factor, so $ord_2(3^n - 1) = 3$ when $\frac{n}{2}$ is odd, or when $n = 4k + 2$.

Solution 2 by Reid W. Barton (10/MA): Use $\Leftrightarrow$ to represent *if and only if*. Note that
$$(ord_2(n) = k) \Leftrightarrow 2^k \text{ divides n and } 2^{k+1} \text{ does not divide n} \Leftrightarrow n \equiv 2^k(mod 2^{k+1}) \quad .$$
    (a.) Answer: *n* odd.

We have $ord_2(3^n - 1) = 1 \Leftrightarrow 3^n - 1 \equiv 2(mod 4) \Leftrightarrow 3^n \equiv 3(mod 4)$. Now $3^1 \equiv 3(mod 4)$ and $3^2 \equiv 1(mod 4)$, so $3^n \equiv 3(mod 4)$ if and only if n is odd.

    (b.) Answer: no *n*.

Since $ord_2(3^n - 1) = 2 \Leftrightarrow 3^n - 1 \equiv 4(mod 8) \Leftrightarrow 3^n \equiv 5(mod 8)$, and $3^1 \equiv 3(mod 8)$, $3^2 \equiv 1(mod 8)$, there is no *n* for which $3^n \equiv 5(mod 8)$, and therefore no *n* for which $ord_2(3^n - 1) = 2$.

    (c.) Answer: $n \equiv 2(mod 4)$.

in this case, $ord_2(3^n - 1) = 3 \Leftrightarrow 3^n - 1 \equiv 8(mod 16) \Leftrightarrow 3^n \equiv 9(mod 16)$, and

$3^1 \equiv 3(mod 16)$, $3^2 \equiv 9(mod 16)$, $3^3 \equiv 11(mod 16)$, and $3^4 \equiv 1(mod 16)$, so $3^n \equiv 9(mod 8)$ if and only if $n \equiv 2(mod 4)$.

Editor's comment: This problem was posed by Dr. Erin Schram of the National Security Agency. Dr. Schram contributed one problem for each round of Year 10. His involvement is greatly appreciated.

3/4/10. Let f be a polynomial of degree 98, such that $f(k) = \dfrac{1}{k}$ for $k = 1, 2, 3, ..., 99$. Determine $f(100)$.

Solution 1 by Wei-Han Liu (8/TN): Consider the expression $f(x) - \dfrac{1}{x}$. This expression has roots at $1, 2, 3, 4, ..., 99$. However, this is not a polynomial. Next consider the function
$$g(x) = xf(x) - 1. \tag{a}$$
Because $f(x)$ is a polynomial of degree 98, $g(x)$ is a polynomial of degree 99. It has the roots $1, 2, 3, 4, ..., 99$. Therefore it must be of the form
$$g(x) = c(x-1)(x-2)(x-3)...(x-99) \tag{b}$$
where c is a constant. To determine c we find that $g(0) = -1$ from equation (a). So we put 0 in for x in equation (b) and get $g(0) = -c(99!) = -1$, so $c = 1/(99!)$ and
$$g(x) = \frac{(x-1)(x-2)(x-3)...(x-99)}{99!}. \tag{c}$$
We want to find $f(100)$, so we plug 100 into both equation (a) and (c) and get
$$100(f(100)) - 1 = \frac{(100-1)(100-2)(100-3)...(100-99)}{99!}$$
$$100(f(100)) - 1 = 1$$
$$100(f(100)) = 2$$
$$f(100) = \frac{1}{50}$$

Therefore, $f(100) = \dfrac{1}{50}$.

Solution 2 by Daniel Moraseski (11/FL): Generalize by replacing 99 with n. Since $f(k)k = 1$ for all integers k between 1 and n,
$$f(x)x - 1 = A(x-1)(x-2)...(x-n),$$
where A is a constant, is also true for these values of k. Now plug in $x = 0$ and get
$$-1 = An!(-1)^n$$
$$A = \frac{(-1)^n}{n!}.$$
If we set A equal to this, the above equation is also true at $x = 0$. Since it is of degree n and is valid at $n+1$ values, it is true for all x. Now we find $f(n+1)$.
$$f(n+1)(n+1) - 1 = \frac{(-1)^{n-1}}{n!}n! = (-1)^{n-1}$$
$$f(n+1) = \frac{(-1)^{n-1}+1}{n+1}$$

$f(n+1) = 0$ for n even. $f(n+1) = \dfrac{2}{n+1}$ for n odd.

Applied to our specific problem, $n = 99$, so $f(100) = 1/50$.

4/4/10. Let A consist of 16 elements of the set $\{1, 2, 3\ldots, 106\}$, so that no two elements of A differ by 6, 9, 12, 15, 18, or 21. Prove that two elements of A must differ by 3.

Solution 1 by Kasia Kobeszko (12/LA): (Proof by contradiction) Since all the differences forbidden within A (including the difference in question, *3*) are multiples of three, the set A can be separated into numbers coming from three disjoint sources: numbers in the form *3n*, in the form *3n+1*, and in the form *3n+2*, for integer *n*'s. The largest of these sources has *36* elements.

Let us assume that no two members of A differ by *3*.

This, combined with the stated difference restrictions, means that two members of A from any one source must differ by at least 24 to both be eligible for membership in A, or when counting by threes, they must be at least eight apart. That means that the most members that can come from one source is $\lceil (\text{size of source})/8 \rceil \leq \lceil 36/8 \rceil = 5$, where $\lceil w \rceil$ is the ceiling function indicating the smallest integer $\geq w$.

With at most *5* elements from each of three sources, A can have at most *15* elements. This contradicts the definition of A as a 16-element set, and disproves our assumption.

Solution 2 by Emily Kendall (10/IN): Assume temporarily that no two members of A differ by three. Consider all members of A congruent to c (mod 3), where $c = 0, 1, or 2$. Because these members all differ by multiples of three, they must differ by at least 24.

106 can be divided into at most 4 intervals of 24. Since any two consecutive members of A congruent to c (mod 3) must be separated by one of these intervals, there can be at most *5* such members.

Therefore, assuming that no two members differ by three, there are at most five members of A congruent to *1* (mod 3), five congruent to *2* (mod 3), and five congruent to *0* (mod 3), for a total of *15*. So in order to have *16* elements in set A, two of them must differ by three.

Solution 3 by Vikki Kowalski (11/AR): What follows is an attempt to construct a set of 16 elements that meet all the required conditions of set A *and* have no two elements which differ by 3.

That would mean that for each new element added to A, 14 other potential elements are made invalid for consideration to become part of the set A. The maximum number of times that a single element could be counted ("overlap" between the set that one member of A makes invalid and the set that another member of A makes invalid) is twice.

This is because in order for an element to be removed from consideration twice, it must be *3n* (Let us define *n* and all integers n_i as integers such that $-8 < n < 8$) greater than a member of A and *3n* units less than a member of A. If an element was removed from consideration three times, this would mean that there must have been an element chosen for set A that was $3n_1$ from it and one $3n_2$ from it in the same direction (since there are only 2 directions and 3 elements, pigeonhole principle). However, this must mean that they are $3(n_1 - n_2)$ units apart, and $-8 < n_1 - n_2 < 8$, so they could not actually both be members of the set A.

The only problem that remains is the elements near the upper and lower limits of the set. If items for set A are selected sufficiently close to the first element in the set, they will each eliminate only *7* elements from consideration for set A. Without loss of generality, we may choose the elements near the lower limit of the set (*1, 2,* and *3*). Selecting these three will cause a total of only *8* elements to be eliminated from consideration (for each one selected). Following these, *3* elements may be selected at *24*-unit intervals (because these three, and each three following, elim-

inate a total of *24* elements). The maximum number which can be chosen for set *A* using this process before all possible elements have been either selected or eliminated from consideration is *15* (this may be repeated five times, terminating with *97, 98*, and *99*).

The only elements which are left to be selected that satisfy the initial constraints (but not the extra one applied originally) will be the ones which have a difference of *3* with one of the elements selected for *A*.

Therefore, two elements of *A* must differ by *3*.

5/4/10. In $\triangle ABC$, let *D, E,* and *F* be the midpoints of the sides of the triangle, and let *P, Q,* and *R* be the midpoints of the corresponding medians, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively, as shown in the figure at the right. Prove that the value of

$$\frac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2}$$

does not depend on the shape of $\triangle ABC$ and find that value.
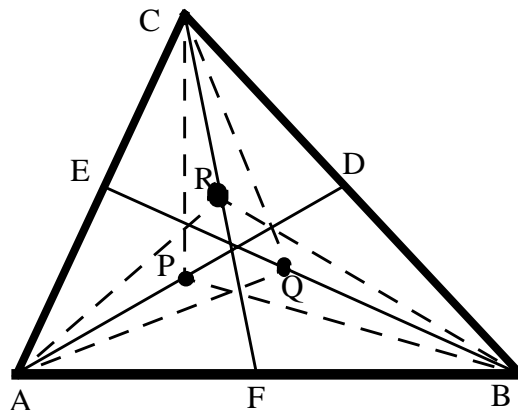
Solution 1 by Luke Gustafson (10/MN):

$\triangle ABC$ can be placed on a coordinate system as shown on the right, with *A* on the origin, *B* at *(4x, 0)*, and *C* at *(4y, 4z)*. By the midpoint theorem, *D = (2x +2y, 2z)*, *E = (2y, 2z), F = (2x, 0), P = (x+y, z), Q = (2x + y, z)*, and *R = (x+2y, 2z)*.

Using the distance formula,
$AQ^2 = (2x+y)^2+z^2 = 4x^2+4xy+ y^2+ z^2$
$AR^2 = (x+2y)^2+4z^2 = x^2+4xy+4y^2+4z^2$
$BP^2=(3x-y)^2+z^2 = 9x^2-6xy+ y^2 + z^2$
$BR^2=(3x-2y)^2+4z^2 = 9x^2-12xy+4y^2+4z^2$
$CP^2=(x-3y)^2+9z^2 = x^2-6xy+9y^2+9z^2$
$CQ^2=(2x-3y)^2+9z^2 = 4x^2-12xy+9y^2+9z^2$.
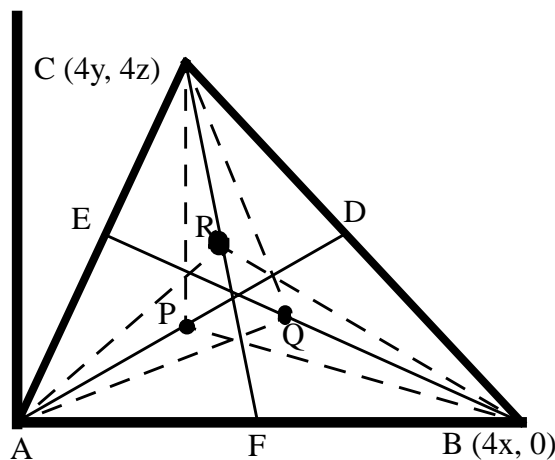Adding these up gives $28(x^2-xy+y^2+z^2)$.
Furthermore,

$AB^2=16x^2$
$BC^2=16x^2-32xy+16y^2+16z^2$
$CA^2=16y^2+16z^2$

Adding these up gives $32(x^2 - xy + y^2 + z^2)$.

That makes $\dfrac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2} = \dfrac{28(x^2 - xy + y^2 + z^2)}{32(x^2 - xy + y^2 + z^2)} = \dfrac{7}{8}$

Since this construction used an arbitrary triangle, this value is independent of the shape of $\triangle ABC$.